

CERTAIN DOUBLE COSET SPACES OF ALGEBRAIC GROUPS AND
RATIONAL BOUNDARY COMPONENTS OF SYMMETRIC BOUNDED DOMAINS

Kenichi IYANAGA

I

In part I we shall consider the problem of determining the order of double cosets $\Gamma \backslash G/P$, where G is a certain k -algebraic group, P is its k -parabolic subgroup and Γ is its arithmetic subgroup. A detailed discussion on the subject is found in [5].

Let k be an algebraic number field of finite degree, and K be either a quadratic extension of k or k itself, and σ the involution of K stabilizing each element of k . Let V be a finite dimensional vector space over K supplied with a non-degenerate k -bilinear form $F: V \times V \rightarrow K$ such that $F(ax, by) = a^\sigma F(x, y)b$ for $a, b \in K$, $x, y \in V$ and that $F(x, y)^\sigma = eF(y, x)$, $e = \pm 1$.

We set $G = \{g \in GL(V); F(g(x), g(y)) = F(x, y), x, y \in V\}$ and $G^1 = G \cap SL(V)$. Then the groups G and G^1 are k -algebraic groups.

Suppose that there exists a proper non-zero subspace W of V such that $F(w, w') = 0$ for all $w, w' \in W$ (i.e. W is a totally isotropic subspace of V). We set $G_W = \{g \in G; g(W) = W\}$. This is a maximal k -parabolic subgroup of G .

Let $\mathcal{O}_K$ be the ring of integers in K and let L be an $\mathcal{O}_K$ -lattice in V . We set $G_L = \{g \in G; g(L) = L\}$. This is an arithmetic subgroup of G .

Similarly, we get a maximal k -parabolic subgroup G_W^1 and an arithmetic subgroup G_L^1 of G^1 .

Now, given any subgroup H of G and $\mathcal{O}_K$ -submodules X, Y of V , we write $X \sim_H Y$ if and only if there exists an element h of H such that $h(X) = Y$.

We denote the set of $\mathcal{O}_K$ -submodules Y such that $X \sim_H Y$ by $(X)_H$. Then, the double coset space $G_L \backslash G / G_W$ is in a bijective correspondence with either one of the sets $(W)_G / \sim_{G_L}$, or $(L)_G / \sim_{G_W}$. Thus the problem of determining the order $|G_L \backslash G / G_W|$ is reduced to a certain classification problem of lattices. The determination of the order $|G_L^1 \backslash G^1 / G_W^1|$ is, to a great extent, reduced to the determination of $|G_L \backslash G / G_W|$.

Associated to the lattice L we have a fractional ideal $\mu_0(L)$ in K , generated by $F(x, y)$ for $x, y \in L$. The lattice L is called a $(\mu_0(L))$ -modular if $L = \{x \in V; F(x, L) \subset \mu_0(L)\}$.

Then we have the following decomposition theorem:

Let L be an $\mathcal{J}$ -modular lattice in V . Then there exist $\mathcal{O}_K$ -ideals $\mathcal{O}_1, \dots, \mathcal{O}_s$, a basis $\{w_1, \dots, w_s\}$ of W , and elements $w'_1, \dots, w'_s$ of V such that

$$L = \sum_{i=1}^s (\mathcal{O}_i^{-\sigma} w_i + \mathcal{O}_i w'_i) + L', \text{ where } \mathcal{O}_1 \supset \mathcal{O}_2 \supset \dots \supset \mathcal{O}_s,$$

$$w_i \in L, F(w_i, w_j) = \delta_{ij}, F(w'_i, w'_j) = m_i \delta_{ij} \text{ for all } i, j.$$

In the above, when $m_i = 0$ for all i (e.g. when $e = -1$), it is easy to determine the order $|G_L \backslash G / G_W|$. When $e = 1$, it becomes necessary to investigate the properties of the submodule $S(\mathcal{O}_K) = \{N(x) + \text{Tr}(y); x, y \in \mathcal{O}_K\}$ of $\mathcal{O}_K$, and submodule $S(L, W, \mathcal{O}) = \{F(ax, ax) + \text{Tr}(b); a \in \mathcal{O}^{-1}, x \in L, b \in \mathcal{O}^{-1-\sigma}\}$ of the module $S(L, \mathcal{O}) = \{F(ax, ax) + \text{Tr}(b); a \in \mathcal{O}^{-1}, x \in L, b \in \mathcal{O}^{-1-\sigma}\}$ for $\mathcal{O}_K$ -ideals $\mathcal{O}$. It can be shown that if K is a quadratic extension of k , then $S(\mathcal{O}_K) = \mathcal{O}_K$, and that the order $|S(L, \mathcal{O}) / S(L, W, \mathcal{O})|$ is generally independent of the choice of the ideal $\mathcal{O}$; we denote the order by $s(L, W)$.

The order $|G_L \backslash G / G_W|$ for an $\mathcal{J}$ -modular lattice L can be evaluated in terms of $h(K)$ (= the class number of K), $h(L')$ (= G -class number of L'), $s(L, W)$ etc. Specifically, we have the following estimation:

1) When $K = k$ and $e = -1$, then $|G_L \backslash G / G_W| = h(k)$.

2) If $S(\mathcal{O}_K) = \mathcal{O}_k$, and $s(M, W) = 1$ for all M belonging to the same G -genus as L , then $|G_L \backslash G/G_W| \leq h(K)h(L')$, and if, moreover, all $\mathcal{J}$ -modular lattices in V are G -equivalent, then $|G_L \backslash G/G_W| = h(K)h(L')$.

The latter case occurs, for example, in the following situations:

- 1) $K = k$, $\dim V$ is odd, $S(\mathcal{O}_K) = \mathcal{O}_k$, $h(k) = 1$,
- 2) K is a quadratic extension of k , $\dim_K V$ is odd, and every ideal class in K is represented by a σ -invariant ideal.

EXAMPLES:

- 1) $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{-1})$, $\dim_K V$ is odd and V has a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ such that $(F(\mathbf{v}_i, \mathbf{v}_j)) = \text{diag.}(1_p, -1_q)$, and $L = \sum \mathcal{O}_K \mathbf{v}_i$. In this case,

$$|G_L \backslash G/G_W| = h(L') \leq |G_L^1 \backslash G^1/G_W^1| \leq 2h(L'),$$

$$h(L') = \begin{cases} 1 & \text{when } W^\perp/W \text{ is indefinite } ([q]), \text{ or the rank of } L' < 5 \text{ [4]} \\ > 1 & \text{when the rank of } L' \geq 5, \\ = 2 & \text{when the rank of } L' = 5, \\ = 4 & \text{when the rank of } L' = 7. \end{cases}$$

- 2) $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{-p})$, $p \equiv 3 \pmod{4}$, $\dim_K V$ is odd and V has a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ such that $(F(\mathbf{v}_i, \mathbf{v}_j)) = \text{diag.}(1_{n-1}, -1)$, and $L = \sum \mathcal{O}_K \mathbf{v}_i$. Then

$$|G_L \backslash G/G_W| = |G_L^1 \backslash G^1/G_W^1| = h(K).$$

II

We assume that G^1 is simply connected (hence, G^1 is either $SU(V, H)$ or $Sp(V, A)$). We assume further that the Lie group $(\mathcal{R}_{k/\mathbb{Q}}(G^1))_{\mathbb{R}}$ admits a maximal compact subgroup $\mathcal{K}$ such that $D = (\mathcal{R}_{k/\mathbb{Q}}(G^1))_{\mathbb{R}}/\mathcal{K}$ has the structure of a symmetric bounded domain (hence, k is totally real, and K is either k itself or a totally imaginary quadratic extension of k).

In this case, the subspace W corresponds to a rational boundary component $B(W)$ of $\bar{D}$, and conversely, for any rational boundary component of $\bar{D}$ there exists a totally isotropic subspace W' of V such that the boundary component may be written as $B(W')$ (cf. [1]); the dimension of such a subspace W' is determined by the given boundary component which we shall call the type of the boundary component. Let $\tilde{B}(W)$ be the set of rational boundary components of $\bar{D}$ having the same type as $B(W)$. $\tilde{B}(W)$ is a G^1 -orbit space. The double coset space $G_L^1 \backslash G^1 / G_W^1$ is in a bijective correspondence with the set of G_L^1 -orbits among $\tilde{B}(W)$.

III

^{may make}
We ~~give~~ a remark concerning our previous work in [2] and [3].

Let $D^* = D \cup \{\text{rational boundary components of } D\}$ supplied with Satake topology, and let $V^* = G_L^1 \backslash D^*$. Then V^* has the structure of a projective variety.

Consider a functor sending the category of Hermitian vector spaces (V, H) to the category of alternating vector spaces (V', A) , where $V' = \mathcal{R}_{K/k} V$ and A is the "imaginary part" of H . This functor naturally induces a rational homomorphism sending $G^1 = SU(V, H)$ into $G' = Sp(V', A)$; lattices L in V naturally correspond to lattices L' in V' .

When L is modular and $\mathcal{M}_0(L)$ is an ideal in k , then the corresponding lattice L' is maximal in V' . When, in general, L is $\mathcal{J}$ -modular, the elementary divisors of L' may be explicitly described in terms of $\mathcal{J}$ if (2) is a prime ideal in k (cf. [6]).

Let D, D' be the symmetric bounded domains corresponding to G^1, G' . Assume that $(\mathcal{R}_{K/Q} \rho)(K) \subset K'$, then ρ induces a holomorphic imbedding of D into D' (cf. [7]); this ρ further induces a morphism of the variety

42

(cf [8])

V^* into V'^* . (We have $\rho(G_L^1) \subset G_L^1$.)

We may ask here, when ^{are} automorphic forms on D with respect to G_L ~~may be~~ extendable to automorphic forms on D' with respect to G_L^1 ? The above I, II may be helpful ⁱⁿ considering this problem.

In particular, the field of rational functions $C(V^*)$, which is identified with the field of automorphic functions on D with respect to G_L^1 , may be identified with a subfield of $C(\rho(V^*))$, and their relations may be described in terms of certain Galois cohomology group (cf. [2], [3]).

Especially, when $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{-p})$, $p \equiv 3 \pmod{4}$, $p > 3$, $\dim_K V$ is odd then $C(V^*) = C(\rho(V^*))$.

REFERENCES

- 1) W.L.Baily, Jr. and A.Borel, Compactification of arithmetic quotients of bounded symmetric domains, Ann. of Math. (3), 84 (1966), 442-528.
- 2) K.Iyanaga, Arithmetic of special unitary groups and their symplectic representations, J. Fac. Sci. Univ. Tokyo, 15 (1968), 35-69.
- 3) _____, Symplectic 表現と保型関数, ^{京大}数理解析研究所講究録 41, 86-55
- 4) _____, Class numbers of definite Hermitian forms, J. Math. Soc. Japan, 21 (1969), 359-374.
- 5) _____, On certain double coset spaces of algebraic groups, J. Math. Soc. Japan, 23 (1971), 103-122.
- 6) _____, Arithmetic of Hermitian forms and related forms, to appear.
- 7) I.Satake, Holomorphic imbeddings of symmetric domains into a Siegel space, Amer.J.Math., 87 (1964), 425-461.
- 8) _____, A note on holomorphic imbeddings and compactifications of symmetric domains, Amer.J.Math. 90 (1968), 231-247.
- 9) G.Shimura, Arithmetic of unitary groups, Ann. of Math., (2) 79 (1964), 369-409.