

REDUCIBILITY OF FUCHSIAN SYSTEMS

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Definition. A system of linear ordinary differential equations with rational coefficient:

$$(1) \quad x' = Ax$$

is reducible, if and only if there is a non-singular linear transformation:

$$x = T(t)y$$

such that the transformed system

$$y' = By = (T^{-1}AT - T^{-1}T')y$$

has a reducible coefficient $B(t)$. A rational transformation $B(T)$ is reducible if it has a proper non-trivial invariant subspace V of C_n independent of t :

$$B(t)V \subset V \quad \text{for all } t.$$

Of course, there is a constant non-singular matrix C , such that $C^{-1}B(t)C$ has a off diagonal block with all the element zero:

$$(2) \quad C^{-1}B(t)C = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$$

if $B(t)$ is reducible.

Definition. Let $S = [\lambda_1, \dots, \lambda_n, \infty]$ be the set of poles of $A(t)$. Let $X(t)$ be some fixed fundamental set of solutions of the system (1). Let γ be a closed circuit on $\mathbb{P}^1 - S$, and let $X(t)M(\gamma)$ be the result of analytic continuation of $X(t)$ along γ . We call the representation of $\pi_1(\mathbb{P}^1 - S)$ in $GL(n, \mathbb{C})$ defined by $\gamma \rightarrow M(\gamma)$, the monodromy representation of (1) with respect to $X(t)$.

Definition. A linear representation of a group G is reducible if there is a proper non-trivial invariant subspace for all the elements.

Theorem 1. If the system (1) is reducible, then every monodromy representation is reducible.

Corollary. If a monodromy representation is irreducible for (1), then (1) is irreducible. (= not reducible).

Proof of the theorem. By a rational transformation, we get a new system of the form:

$$y_1' = B_{11}y_1$$

$$y_2' = B_{21}y_1 + B_{22}y_2$$

We have a non-trivial set of solutions for which $y_1=0$ identically, that is, we have a fundamental set of solutions of the form:

$$Y(t) = \begin{pmatrix} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{pmatrix}$$

If Y_{11} be an r by r matrix, Y_{21} be $(n-r)$ by r , Y_{22} be $(n-r)$ by $(n-r)$, and the zero block be r by $(n-r)$. Then it is clear that the representation with respect to this set has the form:

$$\begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix}$$

Any vector whose first r components are zero is transformed by this type ^{of} matrix from the right into a vector whose first r components are zero.

Theorem 2. If a monodromy representation of the system (1) has an $(n-1)$ dimensional invariant subspace, and it is Fuchsian, then the system itself is reducible.

Proof. We may assume that the invariant subspace V is the set of vectors whose n -th component ~~is~~ is zero. Let $g_1, \dots, g_p$ be the generators of the representation. They have the form:

$$g_j = \left(\begin{array}{c|c} * & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline * & e^{-2\pi i C_j} \end{array} \right)$$

where c_j ($j=1,2,\dots,n$) are some constants determined up to an integral difference. Let $x_n(t)$ be the n -th column vector of the fundamental set $X(t)$ corresponding to the above representation. Then by simple observation, we see the vector solution $x_n(t)$ is transformed into $x_n(t)\exp(-2\pi i c_j)$ by the circuit around j . Hence the column vector $y(t)=r(t)x_n(t)$ with a scalar multiplier;

$$r(t) = (t-a_1)^{c_1} \dots (t-a_m)^{c_m}$$

is single-valued in the entire complex plane. On the other hand, we have assumed the system to be a Fuchsian system, that is, $y(t)$ is a non-trivial rational function, satisfying the system of differential equations:

$$dy/dt = r'x_n(t) + rx_n'(t) = [r'/r + A]y$$

Let $R(t)$ be any non-singular n by n matrix with rational elements whose n -th column vector is $y(t)$. Then the n -th column of the matrix:

$$dR/dt - A(t)R - r'/r \cdot R$$

is identically zero. Multiply R^{-1} from the left, and we see the matrix

$$R^{-1}[R' - AR] - (r'/r)I$$

has zero n -th column. Consequently, the matrix

$$B(t) = R^{-1}[R' - AR]$$

has the n -th column of the form:

$$b_n(t) = (0, 0, \dots, 0, r'/r)$$

That is, $B(t)$ has an invariant subspace of vectors whose n -th component zero.

Theorem 3. Let

$$A = \begin{pmatrix} -a_1 & & & 1 \\ & \ddots & & \vdots \\ & & 0 & \vdots \\ 0 & & & 1 \\ & & & \vdots \\ & & -a_{n-1} & 1 \\ b_1 & \dots & b_{n-1} & -a_n \end{pmatrix} \quad B = \text{diag}[0, 0, \dots, 0, 1]$$

be two constant n by n matrices. We denote the eigenvalues of the matrix A

by $c_1, c_2, \dots, c_n$. We consider the system of n first order linear differential equations:

$$(*) \quad (t-E)dx/dt = Ax$$

under the conditions:

$$1^\circ. \quad a_j \not\equiv 0 \pmod{1} \quad 2^\circ. \quad c_j \not\equiv 0 \pmod{1} \quad 3^\circ. \quad a_j - a_k \not\equiv 0 \pmod{1}$$

for all j and k .

The system $(*)$ has n singular solutions of the form:

$$x_j(t) = t^{-a_j} \sum_{m=0}^{\infty} g_j(m) t^m \quad (j=1, 2, \dots, n-1)$$

$$x_n(t) = (t-1)^{-a_n} \sum_{m=0}^{\infty} g_n(m) (t-1)^m$$

These solutions constitute a fundamental set $X(t)$ of solutions, with respect to which the monodromy has generators of the form:

$$M_0 = \begin{pmatrix} e_1 & & & q_1(e_1-1) \\ & e_2 & & 0 \\ & & \ddots & \\ 0 & & & e_{n-1} & q_{n-1}(e_{n-1}-1) \\ 0 & 0 & \dots & & 1 \end{pmatrix} \quad (e_j = \exp(2\pi i(-a_j)).)$$

$$M_1 = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ & 0 & & \ddots & \\ & & & & 1 & \\ p_1(e_n-1) & p_2(e_n-1) & \dots & p_{n-1}(e_n-1) & & e_n \end{pmatrix}$$

where $2(n-1)$ constants p_j, q_k are given by the formulae:

$$p_j = -b_j \frac{\Gamma(1-a_j) \cdot \Gamma(a_n) \cdot \prod_{k \neq j, n} \Gamma(1-a_j+a_k)}{\prod_k \Gamma(1-a_j+c_k)}$$

$$q_j = -\frac{1}{b_j} \frac{\Gamma(a_j) \cdot \Gamma(1-a_n) \cdot \prod_{k \neq j, n} \Gamma(a_j-a_k)}{\prod_k \Gamma(c_k-a_j)}$$

Theorem 4. Under the conditions of the preceding theorem, the system (*) is irreducible.

Proof. We remark that none of the quantities $p_1, \dots, p_{n-1}, q_1, \dots, q_{n-1}$ is zero. Suppose V is a non-trivial proper linear subspace of C^n such that

$$VM_0 \subset V, \quad VM_1 \subset V$$

There is at least one vector v in V such that the n -th component is not zero.

For if $v = (v_1, \dots, v_{n-1}, 0)$ be a non-trivial vector in V , then from the condition

$vM_0 \in V$, we have;

$$(e_1 v_1, e_2 v_2, \dots, e_{n-1} v_{n-1}, \sum_{j=1}^{n-1} q_j (e_j - 1) v_j)$$

If V consists of the vectors whose n -th component is zero, then we have

$$(e_1 v_1, \dots, e_{n-1} v_{n-1}, 0) \in V$$

$$\sum_{j=1}^{n-1} q_j (e_j - 1) v_j = 0$$

Similarly, we have $(n-1)$ conditions of the form:

$$\sum_{j=1}^{n-1} q_j (e_j - 1) e_j^k v_j = 0 \quad (k=1, \dots, n-1)$$

Since the Vandermode determinant $\det(e_j^k)$ is not zero under the condition 3° , we have identically:

$$q_j (e_j - 1) v_j = 0 \quad j: 1, 2, \dots, n-1.$$

That is, v is a trivial vector.

Suppose now $v = (v_1, \dots, v_{n-1}, 1)$ is in V . Then we have

$$(e_n - 1)^{-1} v(M_1 - I) = (p_1, \dots, p_{n-1}, 1) \in V.$$

We claim that n row vectors $P = (p_1, \dots, p_{n-1}, 1), PM_0, PM_0^2, \dots, PM_0^{n-1}$ are linearly independent, that is, V is actually the full space. If we write the matrix M_0 in the form:

$$M_0 = (I + C)^{-1} \text{diag.}[e_1, \dots, e_{n-1}, 1](I + C)$$

with:

$$C = \begin{array}{cccc} 0 & 0 & \dots & q_1 \\ 0 & 0 & \dots & q_2 \\ & & \dots & \\ 0 & 0 & \dots & 0 \end{array}$$

it is easy to see the k -th power M_0^k to be of the form:

$$M_0^k = \begin{pmatrix} e_1^k & \dots & q_1(e_1^k - 1) \\ \dots & e_2^k & \dots & q_2(e_2^k - 1) \\ \dots & \dots & e_{n-1}^k & q_{n-1}(e_{n-1}^k - 1) \\ \dots & \dots & \dots & 1 \end{pmatrix}$$

Now it is sufficient to show that the following determinant is not zero to prove the theorem:

$$\det \begin{pmatrix} P \\ PM_0 \\ \vdots \\ PM_0^{n-1} \end{pmatrix} = \det \begin{pmatrix} p_1 & p_2 & \dots & \sum p_j q_j (e_j^1 - 1) + 1 \\ p_1 e_1 & p_2 e_2 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ p_1 e_1^{n-1} & p_2 e_2^{n-1} & \dots & \sum p_j q_j (e_j^{n-1} - 1) + 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} p_1 & p_2 & \dots & 1 - \sum p_j q_j \\ p_1 e_1 & p_2 e_2 & \dots & 1 - \sum p_j q_j \\ \vdots & \vdots & \ddots & \vdots \\ p_1 e_1^{n-1} & p_2 e_2^{n-1} & \dots & 1 - \sum p_j q_j \end{pmatrix}$$

$$= \det \begin{pmatrix} e_1^{-1} & e_2^{-1} & \dots & e_{n-1}^{-1} \\ e_1^{2-1} & e_2^{2-1} & \dots & e_{n-1}^{2-1} \\ \vdots & \vdots & \ddots & \vdots \\ e_1^{n-1-1} & e_2^{n-1-1} & \dots & e_{n-1}^{n-1-1} \end{pmatrix} \cdot p_1 p_2 \dots p_{n-1} [1 - \sum p_j q_j]$$

$$= [1 - \sum p_j q_j] p_1 p_2 \dots p_{n-1} (e_1^{-1}) \dots (e_{n-1}^{-1}) V(e_1, \dots, e_{n-1})$$

where $V(e_1, \dots, e_{n-1})$ denotes ~~the~~ Vandermode determinant formed by $n-1$ quantities $e_1, \dots, e_{n-1}$. If we use the formula :

$$1 - \sum_{j=1}^{n-1} p_j q_j = \prod_{j=1}^n [\sin \pi c_j] / [\sin \pi a_j]$$

with the conditions $1^\circ, 2^\circ, 3^\circ$, we see the determinant in question is not zero.

Theorem 5. Let B a diagonal matrix $B = \text{diag}(\lambda_1, \dots, \lambda_n)$ with mutually distinct diagonal elements $\lambda_1, \dots, \lambda_n$. Let A be a matrix whose (j, k) elements is denoted by $a_{j,k}$ with eigenvalues $p_1, p_2, \dots, p_n$. We assume:

$$1^\circ. a_{jj} \neq 0 \pmod{1}, \quad 2^\circ \exists p_k \neq 0 \pmod{1}, \quad 3^\circ p_k - a_{jj} \neq 0 \pmod{1}$$

for all j, k . Then the system

$$(t-B)dx/dt = Ax$$

has a set of n solutions:

$$x_j(t) = (t - \lambda_j)^{a_{jj}} \sum_{m=0}^{\infty} g_j(m) (t - \lambda_j)^m \quad (j=1, 2, \dots, n)$$

which constitute a fundamental set of solutions $X(t)$. The monodromy representation with respect to $X(t)$ has the set of generators:

$$M_j = I + (e_j - 1) \begin{pmatrix} 0 & 0 & \dots & 0 \\ & \dots & \dots & \\ p_{j1} & p_{j2} & \dots, 1, \dots & p_{jn} \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad (j=1, 2, \dots, n)$$

Theorem 6. Besides the 3 conditions in Theorem 5, we assume: there is a number j such that $4^\circ p_{j,k} \neq 0$ for all k , $5^\circ p_{k,j} \neq 0$ for all k , 6° the set of n vectors $P_k = (p_{k1}, p_{k2}, \dots, p_{kn})$, $(k=1, 2, \dots, n)$ are linearly independent. Then the system is irreducible.

Proof. It can be shown as in the proof of theorem 4, that there is at least one vector v whose j -th component is not zero in any invariant subspace V of C_n . Let this non-zero component be 1. Then we have $vM_j - v = (e_j - 1)P_j$. That is to say P_j is in V . Similarly, we have $P_j M_k = P_j + p_{j,k}(e_k - 1)P_k$, and this shows P_k is in V . Now the invariant subspace V contains n linearly independent vectors. This shows that there is no non-trivial proper linear subspace V invariant under the monodromy. This completes the proof.